

FROM UNIPOTENT CLASSES TO CONJUGACY CLASSES IN THE WEYL GROUP

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INTRODUCTION

0.1. Let G be a connected reductive algebraic group over an algebraically closed field \mathbf{k} of characteristic $p \geq 0$. Let $\underline{\underline{G}}$ be the set of unipotent conjugacy classes in G . Let $\underline{\mathbf{W}}$ be the set of conjugacy classes in the Weyl group \mathbf{W} of G . Let $\Phi : \underline{\mathbf{W}} \rightarrow \underline{\underline{G}}$ be the (surjective) map defined in [L2]. For $C \in \underline{\mathbf{W}}$ we denote by m_C the dimension of the fixed point space of $w : V \rightarrow V$ where $w \in C$ and V is the reflection representation of the Coxeter group \mathbf{W} . The following result is an attempt to construct a one sided inverse for Φ .

Theorem 0.2. *Assume that either p is not a bad prime for G or that G is a simple exceptional group. Then for any $\gamma \in \underline{\underline{G}}$ the function $\Phi^{-1}(\gamma) \rightarrow \mathbf{N}$, $C \mapsto m_C$ reaches its minimum at a unique element $C_0 \in \Phi^{-1}(\gamma)$. Thus we have a well defined map $\Psi : \underline{\underline{G}} \rightarrow \underline{\mathbf{W}}$, $\gamma \mapsto C_0$ such that $\Phi\Psi : \underline{\underline{G}} \rightarrow \underline{\underline{G}}$ is the identity map.*

It is likely that (i) the theorem holds without any assumption and (ii) when $p = 0$, the map Ψ coincides with the map defined in [KL, Section 9] (note that the last map has not been computed explicitly in all cases). It is enough to prove the theorem in the case where G is almost simple; moreover in that case it is enough to consider one group in each isogeny class. When G has type A , the theorem is trivial since Φ is a bijection. For G of type B, C, D the proof is given in Section 1. For G of exceptional type the proof is given in Section 2.

1. TYPE B, C, D

1.1. Let \mathcal{P}^1 be the set of sequences $c_* = (c_1 \geq c_2 \geq \cdots \geq c_m)$ in $\mathbf{Z}_{\geq 0}$. For $c_* \in \mathcal{P}^1$ we set $|c_*| = c_1 + c_2 + \cdots + c_\tau$, $\tau_{c_*} = m$. For $N \in \mathbf{N}$ let $\mathcal{P}_N^1 = \{c_* \in \mathcal{P}^1; |c_*| = N\}$.

Let $\tilde{\mathcal{P}} = \{p_* \in \mathcal{P}^1; \tau_{p_*} \in 2\mathbf{N}; p_1 = p_2, p_3 = p_4, \dots\}$; for $m \in \mathbf{N}$ we set $\tilde{\mathcal{P}}_{2m} = \tilde{\mathcal{P}} \cap \mathcal{P}_{2m}^1$.

Let \mathcal{T} be the set of $c_* = (c_1 \geq c_2 \geq \cdots \geq c_\tau) \in \mathcal{P}^1$ such that for any odd j , $|\{k \in [1, \tau]; c_k = j\}|$ is even; for $m \in \mathbf{N}$ let $\mathcal{T}_{2m} = \mathcal{T} \cap \mathcal{P}_{2m}^1$.

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Let \mathcal{S} be the set of $r_* = (r_1 \geq r_2 \geq \cdots \geq r_\sigma) \in \mathcal{P}^1$ such that $r_k \in 2\mathbf{Z}_{>0}$ for all r ; for $m \in \mathbf{N}$ let $\mathcal{S}_{2m} = \mathcal{S} \cap \mathcal{P}_{2m}^1$.

In this subsection we fix $\mathbf{n} \in 2\mathbf{N}$. Let $A'_\mathbf{n}$ be the set of all pairs $(r_*, p_*) \in \mathcal{S} \times \tilde{\mathcal{P}}$ such that $|r_*| + |p_*| = \mathbf{n}$. We define $\iota : A'_\mathbf{n} \rightarrow \mathcal{T}_\mathbf{n}$ by $(r_*, p_*) \mapsto c_*$ where the multiset of entries of c_* is the union of the multiset of entries of r_* with the multiset of entries of p_* . Let $c_* = (c_1 \geq c_2 \geq \cdots \geq c_\tau) \in \mathcal{T}_\mathbf{n}$. We associate to c_* an element $(r_*, p_*) \in \mathcal{S} \times \tilde{\mathcal{P}}$ by specifying the number of times M_e (resp. N_e) that an integer $e \geq 1$ appears in r_* (resp. p_*). Let Q_e be the number of times that e appears as an entry of c_* . If $e \in 2\mathbf{N} + 1$ then $M_e = 0$, $N_e = Q_e$. If $e \in 2\mathbf{N} + 2$ then $M_e = Q_e$, $N_e = 0$. Clearly, $c_* \mapsto (r_*, p_*)$ is a well defined map $\iota' : \mathcal{T}_\mathbf{n} \rightarrow A'_\mathbf{n}$; moreover, $\iota' : \mathcal{T}_\mathbf{n} \rightarrow \mathcal{T}_\mathbf{n}$ is the identity map.

We preserve the notation for c_*, r_*, p_* as above (so that $(r_*, p_*) \in \iota^{-1}(c_*)$) and we assume that $(r'_*, p'_*) \in \iota^{-1}(c_*)$. Let M'_e (resp. N'_e) be the number of times that an integer $e \geq 1$ appears in r'_* (resp. p'_*). Note that $M'_e + N'_e = M_e + N_e$. If $e \in 2\mathbf{N} + 1$ then $M'_e = 0$ hence $N'_e = Q_e$. If $e \in 2\mathbf{N} + 2$ then $N'_e \geq 0$. We see that in all cases we have $N'_e \geq N_e$. It follows that $\sum_e N'_e \geq \sum_e N_e$ (and the equality implies that $N'_e = N_e$ for all e hence $(r'_*, p'_*) = (r_*, p_*)$). We see that

(a) *for any $c_* \in \mathcal{T}_\mathbf{n}$ there is exactly one element $(r_*, p_*) \in \iota^{-1}(c_*)$ such that the number of entries of p_* is minimal (that element is $\iota'(c_*)$).*

1.2. Let $\mathcal{P}^0 = \{p_* \in \mathcal{P}^1; \tau_{p_*} = \text{even}\}$; for $n \in \mathbf{N}$ we set $\mathcal{P}_n^0 = \mathcal{P}^0 \cap \mathcal{P}_n^1$.

Let \mathcal{Q} be the set of all $c_* = (c_1 \geq c_2 \geq \cdots \geq c_\tau) \in \mathcal{P}^1$ such that for any even j , $|\{k \in [1, \tau]; c_k = j\}|$ is even; for $\mathbf{n} \in \mathbf{N}$ let $\mathcal{Q}_\mathbf{n} = \mathcal{Q} \cap \mathcal{P}_\mathbf{n}^1$.

Let \mathcal{R} be the set of all $r_* = (r_1 \geq r_2 \geq \cdots \geq r_\tau) \in \mathcal{Q}$ such that the following conditions are satisfied. Let $J_{r_*} = \{k \in [1, \tau]; r_k \text{ is odd}\}$. We write the multiset $\{r_k; k \in J_{r_*}\}$ as a sequence $r^1 \geq r^2 \geq \cdots \geq r^s$. (We have necessarily $\tau = s \pmod{2}$.) We require that:

- if $\tau \neq 0$ then $1 \in J_{r_*}$;
- if $\tau \neq 0$ is even then $\tau \in J_{r_*}$;
- if $u \in [1, s-1]$ is odd then $r^u > r^{u+1}$;
- if $u \in [1, s-1]$ is even then there is no $k' \in [1, \tau]$ such that $r^u > r_{k'} > r^{u+1}$.

For $\mathbf{n} \in \mathbf{N}$ we set $\mathcal{R}_\mathbf{n} = \mathcal{R} \cap \mathcal{Q}_\mathbf{n}$.

We now fix $\mathbf{n} \in \mathbf{N}$. Define $\kappa \in \{0, 1\}$ and $n \in \mathbf{N}$ by $\mathbf{n} = 2n + \kappa$. Define $\Xi : \mathcal{P}_n^\kappa \rightarrow \mathcal{R}_\mathbf{n}$ by

$$(p_1 \geq p_2 \geq \cdots \geq p_\sigma) \mapsto (2p_1 + \psi(1) \geq 2p_2 + \psi(2) \geq \cdots \geq 2p_\sigma + \psi(\sigma))$$

if $\sigma + \kappa$ is even,

$$(p_1 \geq p_2 \geq \cdots \geq p_\sigma) \mapsto (2p_1 + \psi(1) \geq 2p_2 + \psi(2) \geq \cdots \geq 2p_\sigma + \psi(\sigma) \geq 1)$$

if $\sigma + \kappa$ is odd, where $\psi : [1, \sigma] \rightarrow \{-1, 0, 1\}$ is as follows:

- if $t \in [1, \sigma]$ is odd and $p_t < p_x$ for any $x \in [1, t-1]$ then $\psi(t) = 1$;
- if $t \in [1, \sigma]$ is even and $p_x < p_t$ for any $x \in [t+1, \sigma]$, then $\psi(t) = -1$;

for all other $t \in [1, \sigma]$ we have $\psi(t) = 0$.

Now Ξ is a bijection with inverse map $\Xi' : \mathcal{R}_{\mathbf{n}} \rightarrow \mathcal{P}_{\mathbf{n}}^{\kappa}$ given by

$$(r_1 \geq r_2 \geq \cdots \geq r_{\tau}) \mapsto ((r_1 + \zeta(1))/2 \geq (r_2 + \zeta(2))/2 \geq \cdots \geq (r_{\tau} + \zeta(\tau))/2))$$

if $r_{\tau} > \kappa$,

$$(r_1 \geq r_2 \geq \cdots \geq r_{\tau}) \mapsto ((r_1 + \zeta(1))/2 \geq (r_2 + \zeta(2))/2 \geq \cdots \geq (r_{\tau-1} + \zeta(\tau-1))/2))$$

if $r_{\tau} = \kappa$, where $\zeta : [1, \tau] \rightarrow \{-1, 0, 1\}$ is given by $\zeta(k) = (-1)^k(1 - (-1)^{r_k})/2$. (Thus, $\zeta(k) = (-1)^k$ if r_k is odd and $\zeta(k) = 0$ if r_k is even. We have $r_k + \zeta(k) \in 2\mathbf{N}$ for any k and $r_k + \zeta(k) \geq r_{k+1} + \zeta(k+1)$ for $k \in [1, \tau-1]$.)

Let $A_{\mathbf{n}}$ be the set of all pairs $(r_*, p_*) \in \mathcal{R} \times \tilde{\mathcal{P}}$ such that $|r_*| + |p_*| = \mathbf{n}$. We define $\iota : A_{\mathbf{n}} \rightarrow \mathcal{Q}_{\mathbf{n}}$ by $(r_*, p_*) \mapsto c_*$ where the multiset of entries of c_* is the union of the multiset of entries of r_* with the multiset of entries of p_* . Let $c_* = (c_1 \geq c_2 \geq \cdots \geq c_{\tau}) \in \mathcal{Q}_{\mathbf{n}}$. Let $K = \{k \in [1, \tau]; c_k \text{ is odd}\}$. We write the multiset $\{c_k; k \in K\}$ as a sequence $c^1 \geq c^2 \geq \cdots \geq c^t$. (We have necessarily $\tau = \mathbf{n} = t \pmod{2}$.) We associate to c_* an element $(r_*, p_*) \in \mathcal{Q} \times \tilde{\mathcal{P}}$ by specifying the number of times M_e (resp. N_e) that an integer $e \geq 1$ appears in r_* (resp. p_*). Let Q_e be the number of times that e appears as an entry of c_* .

- (i) If $e \in 2\mathbf{N} + 1$ and $Q_e = 2g + 1$ then $M_e = 1, N_e = 2g$.
- (ii) If $e \in 2\mathbf{N} + 1$ and $Q_e = 2g$, so that $c^d = c^{d+1} = \cdots = c^{d+2g-1} = e$ with d even, then $M_e = 2, N_e = 2g - 2$ (if $g > 0$) and $M_e = N_e = 0$ (if $g = 0$).
- (iii) If $e \in 2\mathbf{N} + 1$ and $Q_e = 2g$ so that $c^d = c^{d+1} = \cdots = c^{d+2g-1} = e$ with d odd then $M_e = 0, N_e = 2g$.

Thus the odd entries of r_* are defined. We write them in a sequence $r^1 \geq r^2 \geq \cdots \geq r^s$.

- (iv) If $e \in 2\mathbf{N} + 2, Q_e = 2g$ and if
 - (*) $r^{2v} > e > r^{2v+1}$ for some v , or $e > r^1$, or $r^s > e$ (with s even),
 then $M_e = 0, N_e = 2g$.

- (v) If $e \in 2\mathbf{N} + 2, Q_e = 2g$ and if (*) does not hold, then $M_e = 2g, N_e = 0$.

Now $r_* \in \mathcal{Q}, p_* \in \tilde{\mathcal{P}}$ are defined and $|r_*| + |p_*| = \mathbf{n}$.

Assume that $|r_*| > 0$; then from (iv) we see that the largest entry of r_* is odd. Assume that $|r_*| > 0$ and \mathbf{n} is even; then from (iv) we see that the smallest entry of r_* is odd. If $u \in [1, s-1]$ and $r^u = r^{u+1}$ then from (i),(ii),(iii) we see that u is even. If $u \in [1, s-1]$ and there is $k' \in [1, \tau]$ such that $r^u > r_{k'} > r^{u+1}$, then $r_{k'}$ is even and $e = r_{k'}$ is as in (v) and u must be odd. We see that $r_* \in \mathcal{R}$.

We see that $c_* \mapsto (r_*, p_*)$ is a well defined map $\iota' : \mathcal{Q}_{\mathbf{n}} \rightarrow A_{\mathbf{n}}$; moreover, $\iota' : \mathcal{Q}_{\mathbf{n}} \rightarrow \mathcal{Q}_{\mathbf{n}}$ is the identity map.

We preserve the notation for c_*, r_*, p_* as above (so that $(r_*, p_*) \in \iota^{-1}(c_*)$) and we assume that $(r'_*, p'_*) \in \iota^{-1}(c_*)$. We write the odd entries of r'_* in a sequence $r'^1 \geq r'^2 \geq \cdots \geq r'^{s'}$.

Let M'_e (resp. N'_e) be the number of times that an integer $e \geq 1$ appears in r'_* (resp. p'_*). Note that $M'_e + N'_e = M_e + N_e$.

In the setup of (i) we have $M'_e = 1, N'_e = N_e$. (Indeed, $M'_e + N'_e$ is odd, N'_e is even hence M'_e is odd. Since M'_e is 0, 1 or 2 we see that it is 1.)

In the setup of (ii) and assuming that $g > 0$ we have $M'_e = 2, N'_e = N_e$ or $M'_e = 0, N'_e = N_e + 2$. (Indeed, $M'_e + N'_e$ is even, N'_e is even hence M'_e is even. Since M'_e is 0, 1 or 2 we see that it is 0 or 2.) If $g = 0$ we have $M'_e = N'_e = 0$.

In the setup of (iii) we have $M'_e = 0, N'_e = N_e$. (Indeed, $M'_e + N'_e$ is even, N'_e is even hence M'_e is even. Since M'_e is 0, 1 or 2 we see that it is 0 or 2. Assume that $M'_e = 2$. Then $e = r'^u = r'^{u+1}$ with u even in $[1, s' - 1]$. We have $c^d = c^{d+1} = \dots = c^{d+2g-1} = e$ with d odd. From the definitions we see that $u = d \pmod 2$ and we have a contradiction. Thus, $M'_e = 0$.)

Now the sequence $r'^1 \geq r'^2 \geq \dots \geq r'^{s'}$ is obtained from the sequence $r^1 \geq r^2 \geq \dots \geq r^s$ by deleting some pairs of the form $r^{2h} = r^{2h+1}$. Hence in the setup of (iv) we have $r^{2v} = r'^{2v'} > e > r'^{2v'+1} = r^{2v+1}$ for some v' or $e > r'^1$ or $r'^{s'} > e$ (with s, s' even) and we see that $M'_e = 0$ so that $N'_e = 2g = N_e$.

In the setup of (v) we have $N'_e \geq 0$.

We see that in all cases we have $N'_e \geq N_e$. It follows that $\sum_e N'_e \geq \sum_e N_e$ (and the equality implies that $N'_e = N_e$ for all e hence $(r'_*, p'_*) = (r_*, p_*)$). We see that

(a) *for any $c_* \in \mathcal{Q}_{\mathbf{n}}$ there is exactly one element $(r_*, p_*) \in \iota^{-1}(c_*)$ such that the number of entries of p_* is minimal (that element is $\iota'(c_*)$).*

Let $\dot{A}_{\mathbf{n}}$ be the set of all pairs $(p'_*, p_*) \in \mathcal{P}^{\kappa} \times \tilde{\mathcal{P}}$ such that $2|p'_*| + 1 + |p_*| = \mathbf{n}$. We have a bijection

(b) $\dot{A}_{\mathbf{n}} \xrightarrow{\sim} A_{\mathbf{n}}, (p'_*, p_*) \mapsto (\Xi(p'_*), p_*)$

where Ξ is as above (with n replaced by $|p'_*|$).

1.3. In this subsection we assume that \mathbf{n} is even. Let $\mathcal{E}_{\mathbf{n}}$ be the set of all $p_* \in \tilde{\mathcal{P}}_{\mathbf{n}}$ such that any entry of p_* is even. We can view $\mathcal{E}_{\mathbf{n}}$ as a subset of $A_{\mathbf{n}}$ by $p_* \mapsto (r_*, p_*)$ where $r_* \in \mathcal{R}_0$ is the empty sequence. Let $\tilde{A}_{\mathbf{n}} = A_{\mathbf{n}} - \mathcal{E}_{\mathbf{n}}$. Moreover, we can view $\tilde{\mathcal{E}}_{\mathbf{n}}$ as a subset of $\mathcal{Q}_{\mathbf{n}}$ by $p_* \mapsto p_*$. Let $\tilde{\mathcal{Q}}_{\mathbf{n}} = \mathcal{Q}_{\mathbf{n}} - \mathcal{E}_{\mathbf{n}}$. Note that ι, ι' restrict to the identity map $\mathcal{E}_{\mathbf{n}} \rightarrow \mathcal{E}_{\mathbf{n}}$. Let $\tilde{\iota} : \tilde{A}_{\mathbf{n}} \rightarrow \tilde{\mathcal{Q}}_{\mathbf{n}}, \tilde{\iota}' : \tilde{\mathcal{Q}}_{\mathbf{n}} \rightarrow \tilde{A}_{\mathbf{n}}$ be the restrictions of ι, ι' . Note that $\tilde{\iota}\tilde{\iota}' : \tilde{\mathcal{Q}}_{\mathbf{n}} \rightarrow \tilde{\mathcal{Q}}_{\mathbf{n}}$ is the identity map.

We can view $\mathcal{E}_{\mathbf{n}}$ as a subset of $\dot{A}_{\mathbf{n}}$ by $p_* \mapsto (p'_*, p_*)$ where p'_* is the empty sequence. Let $\tilde{\dot{A}}_{\mathbf{n}} = \dot{A}_{\mathbf{n}} - \mathcal{E}_{\mathbf{n}}$. The bijection 1.2(b) restricts to a bijection

(a) $\tilde{\dot{A}}_{\mathbf{n}} \xrightarrow{\sim} \tilde{A}_{\mathbf{n}}$.

1.4. In this subsection we assume that $p \neq 2$. Let V be a \mathbf{k} -vector space of finite dimension $\mathbf{n} \geq 3$. Let $\kappa = 0$ if \mathbf{n} is even, $\kappa = 1$ if \mathbf{n} is odd. Let $n = (\mathbf{n} - \kappa)/2$. Let $\epsilon \in \{1, -1\}$. Assume that V has a fixed nonsingular bilinear form $(,) : V \times V \rightarrow \mathbf{k}$ such that $(x, y) = \epsilon(y, x)$ for all $x, y \in V$. Let $Is(V)$ be the group of all isometries of $(,)$ (a closed subgroup of $GL(V)$). We assume that G is the identity component of $Is(V)$.

Assume first that $\epsilon = -1$. We identify $\underline{G} = \mathcal{T}_{\mathbf{n}}$ by associating to $\gamma \in \underline{G}$ the multiset consisting of the sizes of the Jordan blocks of an element of γ . We identify (as in [L2, 1.4, 1.5]) \mathbf{W} with the group W of permutations of $[1, \mathbf{n}]$ commuting

with the involution $i \mapsto \mathbf{n} - i + 1$. We identify $\underline{\mathbf{W}}$ with $A'_\mathbf{n}$ by associating to $C \in \underline{\mathbf{W}}$ (with $w \in C$) the pair (r_*, p_*) where r_* is the multiset consisting of the sizes of cycles of w which commute with the involution above and p_* is the multiset consisting of the sizes of the remaining cycles of w . Using [L2, 3.7, 1.1] we see that the map $\Phi : \underline{\mathbf{W}} \rightarrow \underline{\underline{G}}$ becomes the map $\iota : A'_\mathbf{n} \rightarrow \mathcal{T}_\mathbf{n}$ in 1.1 and 0.2 follows from 1.1(a).

Assume next that $\epsilon = 1$. In the case where \mathbf{n} is even we assume that $\mathbf{n} \geq 8$ and we let $\underline{\underline{G}}_0$ be the set of unipotent classes in G which are also conjugacy classes in $Is(V)$. We identify $\underline{\underline{G}} = \mathcal{Q}_\mathbf{n}$ (if \mathbf{n} is odd) and $\underline{\underline{G}}_0 = \tilde{\mathcal{Q}}_\mathbf{n}$ (if \mathbf{n} is even) by associating to γ in $\underline{\underline{G}}$ or $\underline{\underline{G}}_0$ the multiset consisting of the sizes of the Jordan blocks of an element of γ . If \mathbf{n} is odd we identify (as in [L2, 1.4, 1.5]) $\underline{\mathbf{W}}$ with the group W of permutations of $[1, \mathbf{n}]$ commuting with the involution $i \mapsto \mathbf{n} - i + 1$. If \mathbf{n} is even we identify (as in [L2, 1.4, 1.5]) $\underline{\mathbf{W}}$ with the group W' of even permutations of $[1, \mathbf{n}]$ commuting with the involution $i \mapsto \mathbf{n} - i + 1$; in this case let $\underline{\mathbf{W}}_0$ be the set of conjugacy classes in WW which are also conjugacy classes of the group of all permutations of $[1, \mathbf{n}]$ commuting with the involution above.

We identify $\underline{\mathbf{W}} = \dot{A}_\mathbf{n}$ (if \mathbf{n} is odd) and $\underline{\mathbf{W}}_0 = \tilde{\dot{A}}_\mathbf{n}$ (if \mathbf{n} is even) by associating to C in $\underline{\mathbf{W}}$ or $\underline{\mathbf{W}}_0$ (with $w \in C$) the pair (p'_*, p_*) where p'_* is the multiset consisting of the half sizes of cycles of w (other than fixed points) which commute with the involution above and p_* is the multiset consisting of the sizes of cycles of w which do not commute with the involution above. Using 1.2(b) and 1.3(a) we identify $A_\mathbf{n} = \dot{A}_\mathbf{n}$ if \mathbf{n} is odd and $\tilde{A}_\mathbf{n} = \tilde{\dot{A}}_\mathbf{n}$ if \mathbf{n} is even. Using [L2, 3.8, 3.9, 1.1] we see that the map $\Phi : \underline{\mathbf{W}} \rightarrow \underline{\underline{G}}$ becomes the map $\iota : A_\mathbf{n} \rightarrow \mathcal{Q}_\mathbf{n}$ in 1.2 (if \mathbf{n} is odd) and the map $\Phi : \underline{\mathbf{W}}_0 \rightarrow \underline{\underline{G}}_0$ becomes the map $\tilde{\iota} : \tilde{A}_\mathbf{n} \rightarrow \tilde{\mathcal{Q}}_\mathbf{n}$ in 1.3 (if \mathbf{n} is even) and 0.2 follows from 1.2(a). (Note that if \mathbf{n} is even and $\gamma \in \underline{\underline{G}} - \underline{\underline{G}}_0$ then $\Phi^{-1}(\gamma)$ is a single element hence for such γ the statement of 0.2 is trivial.)

2. EXCEPTIONAL GROUPS

2.1. In 2.2-2.6 we describe explicitly the map $\Phi : \underline{\mathbf{W}} \rightarrow \underline{\underline{G}}$ in the case where G is a simple exceptional group in the form of tables. Each table consists of lines of the form $[a, b, \dots, r] \mapsto s$ where $s \in \underline{\underline{G}}$ is specified by its name in [Sp] and a, b, \dots, r are the elements of $\underline{\mathbf{W}}$ which are mapped by Φ to s (here a, b, \dots, r are specified by their name in [Ca]); by inspection we see that 0.2 holds in each case and in fact $\Psi(s) = a$ is the first element of $\underline{\mathbf{W}}$ in the list a, b, \dots, r . The tables are obtained from the results in [L2].

2.2. Type G_2 . If $p \neq 3$ we have

$$\begin{aligned} [A_0] &\mapsto A_0 \\ [A_1] &\mapsto A_1 \\ [A_1 + \tilde{A}_1, \tilde{A}_1] &\mapsto \tilde{A}_1 \\ [A_2] &\mapsto G_2(a_1) \end{aligned}$$

$$[G_2] \mapsto G_2$$

When $p = 3$ the line $[A_1 + \tilde{A}_1, \tilde{A}_1] \mapsto \tilde{A}_1$ should be replaced by $[A_1 + \tilde{A}_1] \mapsto \tilde{A}_1$, $[\tilde{A}_1] \mapsto (\tilde{A}_1)_3$.

2.3. Type F_4 . If $p \neq 2$ we have

$$\begin{aligned} [A_0] &\mapsto A_0 \\ [A_1] &\mapsto A_1 \\ [2A_1, \tilde{A}_1] &\mapsto \tilde{A}_1 \\ [4A_1, 3A_1, 2A_1 + \tilde{A}_1, A_1 + \tilde{A}_1] &\mapsto A_1 + \tilde{A}_1 \\ [A_2] &\mapsto A_2 \\ [\tilde{A}_2] &\mapsto \tilde{A}_2 \\ [A_2 + \tilde{A}_1] &\mapsto A_2 + \tilde{A}_1 \\ [A_2 + \tilde{A}_2, \tilde{A}_2 + A_1] &\mapsto \tilde{A}_2 + A_1 \\ [A_3, B_2] &\mapsto B_2 \\ [A_3 + \tilde{A}_1, B_2 + A_1] &\mapsto C_3(a_1) \\ [D_4(a_1)] &\mapsto F_4(a_3) \\ [D_4, B_3] &\mapsto B_3 \\ [C_3 + A_1, C_3] &\mapsto C_3 \\ [F_4(a_1)] &\mapsto F_4(a_2) \\ [B_4] &\mapsto F_4(a_1) \\ [F_4] &\mapsto F_4 \end{aligned}$$

When $p = 2$ the lines $[2A_1, \tilde{A}_1] \mapsto \tilde{A}_1$, $[A_2 + \tilde{A}_2, \tilde{A}_2 + A_1] \mapsto \tilde{A}_2 + A_1$, $[A_3, B_2] \mapsto B_2$, $[A_3 + \tilde{A}_1, B_2 + A_1] \mapsto C_3(a_1)$, should be replaced by

$$\begin{aligned} [2A_1] &\mapsto \tilde{A}_1, [\tilde{A}_1] \mapsto (\tilde{A}_1)_2 \\ [A_2 + \tilde{A}_2] &\mapsto \tilde{A}_2 + A_1, [\tilde{A}_2 + A_1] \mapsto (\tilde{A}_2 + A_1)_2 \\ [A_3] &\mapsto B_2, [B_2] \mapsto (B_2)_2 \\ [A_3 + \tilde{A}_1] &\mapsto C_3(a_1), [B_2 + A_1] \mapsto (C_3(a_1))_2 \end{aligned}$$

respectively.

2.4. Type E_6 . We have

$$\begin{aligned} [A_0] &\mapsto A_0 \\ [A_1] &\mapsto A_1 \\ [2A_1] &\mapsto 2A_1 \\ [4A_1, 3A_1] &\mapsto 3A_1 \\ [A_2] &\mapsto A_2 \\ [A_2 + A_1] &\mapsto A_2 + A_1 \\ [2A_2] &\mapsto 2A_2 \\ [A_2 + 2A_1] &\mapsto A_2 + 2A_1 \\ [A_3] &\mapsto A_3 \\ [3A_2, 2A_2 + A_1] &\mapsto 2A_2 + A_1 \\ [A_3 + 2A_1, A_3 + A_1] &\mapsto A_3 + A_1 \\ [D_4(a_1)] &\mapsto D_4(a_1) \\ [A_4] &\mapsto A_4 \end{aligned}$$

$$\begin{aligned}
[D_4] &\mapsto D_4 \\
[A_4 + A_1] &\mapsto A_4 + A_1 \\
[A_5 + A_1, A_5] &\mapsto A_5 \\
[D_5(a_1)] &\mapsto D_5(a_1) \\
[E_6(a_2)] &\mapsto A_5 + A_1 \\
[D_5] &\mapsto D_5 \\
[E_6(a_1)] &\mapsto E_6(a_1) \\
[E_6] &\mapsto E_6
\end{aligned}$$

2.5. Type E_7 . If $p \neq 2$ we have

$$\begin{aligned}
[A_0] &\mapsto A_0 \\
[A_1] &\mapsto A_1 \\
[2A_1] &\mapsto 2A_1 \\
[3A'_1] &\mapsto 3A'_1 \\
[4A''_1, 3A''_1] &\mapsto 3A'_1 \\
[A_2] &\mapsto A_2 \\
[7A_1, 6A_1, 5A_1, 4A'_1] &\mapsto 4A_1 \\
[A_2 + A_1] &\mapsto A_2 + A_1 \\
[A_2 + 2A_1] &\mapsto A_2 + 2A_1 \\
[A_3] &\mapsto A_3 \\
[2A_2] &\mapsto 2A_2 \\
[A_2 + 3A_1] &\mapsto A_2 + 3A_1 \\
[A_3 + A'_1] &\mapsto (A_3 + A_1)'' \\
[3A_2, 2A_2 + A_1] &\mapsto 2A_2 + A_1 \\
[A_3 + 2A''_1, A_3 + A''_1] &\mapsto (A_3 + A_1)' \\
[D_4(a_1)] &\mapsto D_4(a_1) \\
[A_3 + 3A_1, A_3 + 2A'_1] &\mapsto A_3 + 2A_1 \\
[D_4] &\mapsto D_4 \\
[D_4(a_1) + A_1] &\mapsto D_4(a_1) + A_1 \\
[D_4(a_1) + 2A_1, A_3 + A_2] &\mapsto A_3 + A_2 \\
[2A_3 + A_1, A_3 + A_2 + A_1] &\mapsto A_3 + A_2 + A_1 \\
[A_4] &\mapsto A_4 \\
[D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] &\mapsto D_4 + A_1 \\
[A'_5] &\mapsto A'_5 \\
[A_4 + A_1] &\mapsto A_4 + A_1 \\
[D_5(a_1)] &\mapsto D_5(a_1) \\
[A_4 + A_2] &\mapsto A_4 + A_2 \\
[A_5 + A''_1, A'_5] &\mapsto A'_5 \\
[A_5 + A_2, A_5 + A'_1] &\mapsto (A_5 + A_1)'' \\
[D_5(a_1) + A_1] &\mapsto D_5(a_1) + A_1 \\
[E_6(a_2)] &\mapsto (A_5 + A_1)' \\
[D_6(a_2) + A_1, D_6(a_2)] &\mapsto D_6(a_2) \\
[E_7(a_4)] &\mapsto D_6(a_2) + A_1 \\
[D_5] &\mapsto D_5
\end{aligned}$$

$$\begin{aligned}
[A_6] &\mapsto A_6 \\
[D_5 + A_1] &\mapsto D_5 + A_1 \\
[D_6(a_1)] &\mapsto D_6(a_1) \\
[A_7] &\mapsto D_6(a_1) + A_1 \\
[E_6(a_1)] &\mapsto E_6(a_1) \\
[D_6 + A_1, D_6] &\mapsto D_6 \\
[E_6] &\mapsto E_6 \\
[E_7(a_3)] &\mapsto D_6 + A_1 \\
[E_7(a_2)] &\mapsto E_7(a_2) \\
[E_7(a_1)] &\mapsto E_7(a_1) \\
[E_7] &\mapsto E_7
\end{aligned}$$

If $p = 2$, the line $[D_4(a_1) + 2A_1, A_3 + A_2] \mapsto A_3 + A_2$ should be replaced by $[D_4(a_1) + 2A_1] \mapsto A_3 + A_2$, $[A_3 + A_2] \mapsto (A_3 + A_2)_2$.

2.6. Type E_8 . If $p \neq 2, 3$ we have

$$\begin{aligned}
[A_0] &\mapsto A_0 \\
[A_1] &\mapsto A_1 \\
[2A_1] &\mapsto 2A_1 \\
[4A'_1, 3A_1] &\mapsto 3A_1 \\
[A_2] &\mapsto A_2 \\
[8A_1, 7A_1, 6A_1, 5A_1, 4A''_1] &\mapsto 4A_1 \\
[A_2 + A_1] &\mapsto A_2 + A_1 \\
[A_2 + 2A_1] &\mapsto A_2 + 2A_1 \\
[A_3] &\mapsto A_3 \\
[A_2 + 4A_1, A_2 + 3A_1] &\mapsto A_2 + 3A_1 \\
[2A_2] &\mapsto 2A_2 \\
[3A_2, 2A_2 + A_1] &\mapsto 2A_2 + A_1 \\
[A_3 + 2A'_1, A_3 + A_1] &\mapsto A_3 + A_1 \\
[D_4(a_1)] &\mapsto D_4(a_1) \\
[4A_2, 3A_2 + A_1, 2A_2 + 2A_1] &\mapsto 2A_2 + 2A_1 \\
[D_4] &\mapsto D_4 \\
[A_3 + 4A_1, A_3 + 3A_1, A_3 + 2A''_1] &\mapsto A_3 + 2A_1 \\
[D_4(a_1) + A_1] &\mapsto D_4(a_1) + A_1 \\
[2A'_3, A_3 + A_2] &\mapsto A_3 + A_2 \\
[A_4] &\mapsto A_4 \\
[2A_3 + 2A_1, A_3 + A_2 + 2A_1, 2A_3 + A_1, A_3 + A_2 + A_1] &\mapsto A_3 + A_2 + A_1 \\
[D_4(a_1) + A_2] &\mapsto D_4(a_1) + A_2 \\
[D_4 + 4A_1, D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] &\mapsto D_4 + A_1 \\
[2D_4(a_1), D_4(a_1) + A_3, 2A''_3] &\mapsto 2A_3 \\
[A_4 + A_1] &\mapsto A_4 + A_1 \\
[D_5(a_1)] &\mapsto D_5(a_1) \\
[A_4 + 2A_1] &\mapsto A_4 + 2A_1 \\
[A_4 + A_2] &\mapsto A_4 + A_2 \\
[A_4 + A_2 + A_1] &\mapsto A_4 + A_2 + A_1
\end{aligned}$$

$$\begin{aligned}
[D_5(a_1) + A_1] &\mapsto D_5(a_1) + A_1 \\
[A_5 + A'_1, A_5] &\mapsto A_5 \\
[D_4 + A_3, D_4 + A_2] &\mapsto D_4 + A_2 \\
[E_6(a_2)] &\mapsto (A_5 + A_1)'' \\
[2A_4, A_4 + A_3] &\mapsto A_4 + A_3 \\
[D_5] &\mapsto D_5 \\
[D_5(a_1) + A_3, D_5(a_1) + A_2] &\mapsto D_5(a_1) + A_2 \\
[A_5 + A_2 + A_1, A_5 + A_2, A_5 + 2A_1, A_5 + A''_1] &\mapsto (A_5 + A_1)' \\
[E_6(a_2) + A_2, E_6(a_2) + A_1] &\mapsto A_5 + 2A_1 \\
[2D_4, D_6(a_2) + A_1, D_6(a_2)] &\mapsto D_6(a_2) \\
[E_7(a_4) + A_1, E_7(a_4)] &\mapsto A_5 + A_2 \\
[D_5 + 2A_1, D_5 + A_1] &\mapsto D_5 + A_1 \\
[E_8(a_8)] &\mapsto 2A_4 \\
[D_6(a_1)] &\mapsto D_6(a_1) \\
[A_6] &\mapsto A_6 \\
[A_6 + A_1] &\mapsto A_6 + A_1 \\
[A'_7] &\mapsto D_6(a_1) + A_1 \\
[A_7 + A_1, D_5 + A_2] &\mapsto D_5 + A_2 \\
[E_6(a_1)] &\mapsto E_6(a_1) \\
[D_6 + 2A_1, D_6 + A_1, D_6] &\mapsto D_6 \\
[D_7(a_2)] &\mapsto D_7(a_2) \\
[E_6] &\mapsto E_6 \\
[D_8(a_3), A''_7] &\mapsto A_7 \\
[E_6(a_1) + A_1] &\mapsto E_6(a_1) + A_1 \\
[E_7(a_3)] &\mapsto D_6 + A_1 \\
[A_8] &\mapsto D_8(a_3) \\
[D_8(a_2), D_7(a_1)] &\mapsto D_7(a_1) \\
[E_6 + A_2, E_6 + A_1] &\mapsto E_6 + A_1 \\
[E_7(a_2) + A_1, E_7(a_2)] &\mapsto E_7(a_2) \\
[E_8(a_6)] &\mapsto A_8 \\
[E_8(a_7)] &\mapsto E_7(a_2) + A_1 \\
[D_8(a_1), D_7] &\mapsto D_7 \\
[E_8(a_3)] &\mapsto D_8(a_1) \\
[E_7(a_1)] &\mapsto E_7(a_1) \\
[D_8] &\mapsto E_7(a_1) + A_1 \\
[E_8(a_5)] &\mapsto D_8'' \\
[E_7 + A_1, E_7] &\mapsto E_7 \\
[E_8(a_4)] &\mapsto E_7 + A_1'' \\
[E_8(a_2)] &\mapsto E_8(a_2) \\
[E_8(a_1)] &\mapsto E_8(a_1) \\
[E_8] &\mapsto E_8
\end{aligned}$$

If $p = 3$ the line $[D_8(a_3), A''_7] \mapsto A_7$ should be replaced by $[D_8(a_3)] \mapsto A_7$, $[A''_7] \mapsto$

$(A_7)_3$. If $p = 2$ the lines $[2A'_3, A_3 + A_2] \mapsto A_3 + A_2$, $[D_4 + A_3, D_4 + A_2] \mapsto D_4 + A_2$, $[A_7 + A_1, D_5 + A_2] \mapsto D_5 + A_2$, $[D_8(a_2), D_7(a_1)] \mapsto D_7(a_1)$ should be replaced by

$$\begin{aligned} [2A'_3] &\mapsto A_3 + A_2, [A_3 + A_2] \mapsto (A_3 + A_2)_2 \\ [D_4 + A_3] &\mapsto D_4 + A_2, [D_4 + A_2] \mapsto (D_4 + A_2)_2 \\ [A_7 + A_1] &\mapsto D_5 + A_2, [D_5 + A_2] \mapsto (D_5 + A_2)_2 \\ [D_8(a_2)] &\mapsto D_7(a_1), [D_7(a_1)] \mapsto (D_7(a_1))_2 \end{aligned}$$

respectively.

3. RELATION WITH UNIPOTENT PIECES

3.1. Let G' be a connected reductive group over \mathbf{C} of the same type as G . We identify \mathbf{W} with the Weyl group of G' . Let $\underline{\underline{G'}}$ be the set of unipotent classes in G' . In [L1, 6.8] we have defined a partition of the unipotent variety of G into "unipotent pieces" indexed by $\underline{\underline{G'}}$. We define a surjective map $\rho : \underline{\underline{G}} \rightarrow \underline{\underline{G'}}$ by associating to $\gamma \in \underline{\underline{G}}$ the element $\gamma' \in \underline{\underline{G'}}$ which indexes the unipotent piece that contains γ . Let $\pi : \underline{\underline{G'}} \rightarrow \underline{\underline{G}}$ be the map defined in [L2, 4.1]. Note that $\rho\pi = 1$. Let $\Phi' : \mathbf{W} \rightarrow \underline{\underline{G'}}$, $\Psi' : \underline{\underline{G'}} \rightarrow uWW$ be the maps analogous to $\Phi : \mathbf{W} \rightarrow \underline{\underline{G}}$ in [L2, 4.5] and $\Psi : \underline{\underline{G}} \rightarrow \mathbf{W}$ in 0.2. One can show that

$$(a) \quad \Phi' = \rho\Phi$$

and (assuming that Ψ is defined):

$$(b) \quad \Psi' = \Psi\pi.$$

In the case where G is simple of exceptional type this follows from the tables in Section 2.

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